

ROYDEN'S LEMMA IN INFINITE DIMENSIONS AND HILBERT-HARTOGS MANIFOLDS

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ABSTRACT. We prove the Royden's Lemma for complex Hilbert manifolds, *i.e.*, that a holomorphic imbedding of the closure of a finite dimensional, closed, strictly pseudoconvex domain into a complex Hilbert manifold extends to a biholomorphic mapping onto a product of this domain with the unit ball in Hilbert space. This reduces several problems concerning complex Hilbert manifolds to open subsets of a Hilbert space. As an illustration we prove some results on generalized loop spaces of complex manifolds.

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1. INTRODUCTION

Throughout this paper by capital Latin letters, like S, X, Y , we shall denote finite dimensional real and complex manifolds. By calligraphic letters, like $\mathcal{S}, \mathcal{X}, \mathcal{Y}$ - complex Hilbert manifolds. Our Hilbert manifolds are modeled over l^2 . Recall that the Hartogs figure is the following domain in \mathbb{C}^2 :

$$H(r) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < r \text{ or } 1 - r < |z_1| < 1, |z_2| < 1\}. \quad (1.1)$$

Here $r > 0$ is a small positive number, its precise value is usually irrelevant: if something holds for some $0 < r < 1$ then the same usually holds for all other $0 < r' < 1$. We say that a complex Hilbert manifold \mathcal{Y} is Hilbert-Hartogs (or, simply Hartogs) if every holomorphic mapping $f : H(r) \rightarrow \mathcal{Y}$ extends to a holomorphic mapping $\tilde{f} : \Delta^2 \rightarrow \mathcal{Y}$ of the unit bidisk into \mathcal{Y} . Our ultimate goal is the following infinite dimensional version of the classical Continuity Principle in the form of Behnke and Sommer, see [BS]:

Theorem 1. (Continuity Principle) *Let \mathcal{U} be an open subset in a complex Hilbert manifold \mathcal{X} and let \mathcal{Y} be a Hilbert-Hartogs manifold. Suppose we are given a continuous family of analytic disks $\{\varphi_t : \bar{\Delta} \rightarrow \mathcal{X}\}_{t \in [0,1]}$ in \mathcal{X} such that*

- i) $\varphi_0(\bar{\Delta}) \subset \mathcal{U}$;
- ii) $\varphi_t(\partial\Delta) \subset \mathcal{U}$ for all $t \in [0, 1]$.

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Then every holomorphic mapping $f : \mathcal{U} \rightarrow \mathcal{Y}$ analytically extends along $\{\varphi_t(\bar{\Delta})\}$.

By an analytic disk we mean a holomorphic mapping φ of some neighborhood of $\bar{\Delta}$ into \mathcal{X} . Notions of continuity of a family of disks and analytic continuation along such families are self-obvious, see however section 2 for precise definitions. The statement of the Continuity Principle means that Hilbert-Hartogs manifolds have much stronger extension properties than it is postulated in their definition.

The main ingredient of the proof is a version of Royden's Lemma for complex Hilbert manifolds, which we believe is of independent interest.

Theorem 2. *Let $\varphi : \bar{\Delta} \rightarrow \mathcal{X}$ be an imbedded analytic disk in a complex Hilbert manifold. Then there exists an $\varepsilon > 0$ and an extension of φ to a biholomorphic mapping $\tilde{\varphi} : \bar{\Delta} \times \bar{B}^\infty(\varepsilon) \rightarrow \mathcal{X}$.*

Here $\bar{B}^\infty(\varepsilon)$ stands for the closed ball of radius ε in l^2 .

Remark 1. It is worth to mention that in this statement the disk Δ can be replaced by *any* finite dimensional, bounded strictly pseudoconvex domain D . The conclusion will be the same, in particular, because all holomorphic Hilbert bundles over such D are trivial. Let us underline also that both φ and its extension $\tilde{\varphi}$ are defined in a neighborhood of the corresponding closed sets.

This statement allows to reduce some questions, as the proof of the Continuity Principle, from Hilbert manifolds to open subsets in l^2 . The proof follows the "Einsatz of Royden" from [Ro] with only one difference: instead of performing finite number of steps and then using the Stein theory (which we do not have in infinite dimensions), we perform a sequence of steps and prove that this procedure converges and gives a desired extension.

In section 3 we concentrate our attention on loop spaces of complex manifolds and prove that

Theorem 3. *Any generalized loop space of a Hartogs complex manifold is Hilbert-Hartogs.*

This provides us a lot of interesting examples of infinite dimensional Hartogs manifolds and, moreover, shows that some "a priori unknown" compact complex manifolds (like surfaces from the class VII_0^+ of \mathbb{S}^6) if they do exist should be not only Hartogs themselves but also have all their (generalized) loop spaces Hilbert-Hartogs, see section 3 for more details.

2. ROYDEN'S LEMMA AND CONTINUITY PRINCIPLE

2.1. Hilbert manifolds. $\Delta(r)$ denotes the disk of radius r in \mathbb{C} centered at zero, Δ - the unit disk, $\Delta^q(r)$ the polydisk in \mathbb{C}^q of radius r and $B^q(r)$ the ball of radius r in \mathbb{C}^q , B^q - the unit ball, Δ^q - the unit polydisk and A_{r_1, r_2}^q - a spherical shell $B^q(r_2) \setminus \bar{B}^q(r_1)$, $r_1 < r_2$. Here $\|\cdot\|$ stands for the standard Euclidean norm in \mathbb{C}^q .

The notation l^2 throughout this paper stands for the Hilbert space of sequences of complex numbers $z = \{z_k\}_{k=1}^\infty$ such that $\|z\|^2 := \sum_k |z_k|^2 < \infty$ with the standard Hermitian scalar product $(z, w) = \sum_k z_k \bar{w}_k$ and standard basis $\{e_1, e_2, \dots\}$. By $B^\infty(r)$ we denote the ball of radius r in l^2 and by B^∞ the unit ball. The q -dimensional complex linear space \mathbb{C}^q will be often identified with the subspace $l_q^2 = \text{span}\{e_1, \dots, e_q\} \subset l^2$ and we shall often write (with some ambiguity) $l^2 = \mathbb{C}^q \oplus l$, where l is the orthogonal complement to \mathbb{C}^q and use coordinates $z' = (z_1, \dots, z_q)$ for \mathbb{C}^q and $z'' = \{z_{q+1}, z_{q+2}, \dots\}$ for l . Moreover, we

shall think about l as of a copy of l^2 itself and henceforth using notations like $B^q \times B^\infty$ to denote an obvious subset in l^2 .

Hilbert manifolds of this paper are modeled over l^2 , i.e., they are Hausdorff topological spaces locally homeomorphic to open subsets of l^2 with biholomorphic transition mappings. l^2 -valued holomorphic mapping on an open subset D of l^2 is a mapping $f : D \rightarrow l^2$ which is Fréchet differentiable at all points of D or, equivalently, which in a neighborhood of every point $z^0 \in D$ can be represented by a convergent power series

$$f(z) = \sum_{n=0}^{\infty} P_n(z - z^0), \quad (2.1)$$

where P_n are *continuous homogeneous* polynomials of degree n satisfying

$$\limsup_{n \rightarrow \infty} \|P_n\|^{\frac{1}{n}} = \frac{1}{r} < \infty. \quad (2.2)$$

The the radius of convergence $r > 0$, which depends on z^0 , is the supremum of radii of balls $B^\infty(z^0, r')$ centered at z^0 which are contained in D and such that f is bounded on $B^\infty(z^0, r')$. The norm of a continuous n -homogeneous polynomial is defined as

$$\|P\| := \sup \{ \|P(x)\| : \|x\| \leq 1 \}. \quad (2.3)$$

Cauchy-Hadamard formula (2.2) guarantees the uniform and absolute convergence of power series (2.1) on every ball $B(z^0, r')$ with $r' < r$, see [Mu] for more details. Note, however, that r can be smaller then the distance from z^0 to the boundary ∂D of D . It is important to note that the space \mathcal{P}_n of continuous, homogeneous polynomial mappings of degree n from l^2 to itself is a Banach space with respect to the norm (2.3).

Let us quote the following result of G. Henkin.

Theorem 2.1. *Let D be a bounded strictly pseudoconvex domain in \mathbb{C}^q . Then there exists a constant γ_D such that for every smooth $\bar{\partial}$ -closed $(0,1)$ -form f on \bar{D} with coefficients in a Banach space E there exists a smooth function $u : \bar{D} \rightarrow E$ such that*

$$\bar{\partial}u = f, \quad \text{and} \quad \|u\|_\infty \leq \gamma_D \|f\|_\infty. \quad (2.4)$$

In this Theorem if $f = \sum_{k=1}^q f_k d\bar{z}_k$ then $\|f\|_\infty := \sum_{k=1}^q \|f_k\|_\infty$. For the proof we refer to [He] with the remark that it goes through without any changes for Banach-valued forms.

We shall need also a version of Grauert's theorem due to Bungart, see [Bu] and [Le] Theorems 3.2 and 9.2, which, combined with the Theorem of Kuiper about contractibility of the group of invertible operators in infinite dimensional Hilbert space, see [Ku], gives the following:

Theorem 2.2. *Let D be a Stein manifold and \mathcal{E} a holomorphic Hilbert vector bundle over D . Then \mathcal{E} is holomorphically trivial and, moreover, if $\mathcal{U} = \{U_\alpha\}$ is a locally finite Stein covering of D then for every cocycle $f \in Z^1(\mathcal{U}, \mathcal{G})$ there exists a cochain $c \in C^0(\mathcal{U}, \mathcal{G})$ such that $\delta(c) = f$.*

Here a Hilbert bundle is understood as being locally isomorphic to products $U \times l^2$ and \mathcal{G} is a sheaf of holomorphic mappings with values in the group of invertible operators on l^2 .

2.2. Royden's Lemma. We need to prove a version of Royden's lemma in infinite dimensions. By an analytic q -disk in a complex Hilbert manifold \mathcal{X} we understand a holomorphic mapping φ of a neighborhood of a closure of a relatively compact strongly pseudoconvex domain $D \Subset \mathbb{C}^q$ into \mathcal{X} . The image $\varphi(\bar{D})$ we shall denote by Φ . Therefore, by saying that a q -disk $\Phi = \varphi(\bar{D})$ is analytic we mean that φ holomorphically extends onto a neighborhood of \bar{D} . Saying that Φ is imbedded we mean that φ is an imbedding in a neighborhood of \bar{D} .

Theorem 2.3. (Royden's Lemma) *Let $\Phi = \varphi(\bar{D})$ be an imbedded analytic q -disk in a complex Hilbert manifold \mathcal{X} . Then there exists an $\varepsilon > 0$ and a biholomorphic extension $\tilde{\varphi}$ of φ to a neighborhood of $\bar{D} \times \bar{B}^\infty(\varepsilon)$.*

Proof. More precisely this statement means that there exists a neighborhood V of Φ in \mathcal{X} and a neighborhood W of $\bar{D} \times \bar{B}^\infty(\varepsilon)$ in l^2 such that $\tilde{\varphi}: W \rightarrow V$ is a biholomorphism.

Remark 2.1. Let us make clear that in this form Theorem 2.3 holds true only in infinite dimension, because when the manifold in question is finite dimensional the normal bundle to $\Phi = \varphi(\bar{D})$ can be non trivial.

Now let us turn to the proof. We choose $r > 0$ such that φ extends as an imbedding onto an $2r$ -neighborhood of D , i.e., onto $D^{2r} := \{z \in \mathbb{C}^q : d(z, D) < 2r\}$.

Step 1. We shall state this step in the form of a lemma.

Lemma 2.1. *Let $\varphi : B^q(a, \varepsilon) \rightarrow \mathcal{X}$ be a holomorphic map of a small ball centered at $a \in \mathbb{C}^q$ into a complex Hilbert manifold \mathcal{X} with $\varphi(a) = b \in \mathcal{X}$, such that $d_a\varphi : \mathbb{C}^q \rightarrow T_b\mathcal{X}$ is injective. Then one can find a coordinate chart (V, h) in a neighborhood of b such that:*

- i) *V is mapped by h onto a neighborhood V' of the point $(a, 0) \in l^2$ with $h(b) = (a, 0)$, and if $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_q), \tilde{w} = (\tilde{w}_{q+1}, \dots)$ are standard coordinates in l^2 then $V' = \{(\tilde{z}, \tilde{w}) \in l^2 : \|\tilde{z} - a\| < \delta, \|\tilde{w}\| < \delta\}$ for an appropriate $\delta > 0$;*
- ii) *the map φ from $U := \varphi^{-1}(V)$ to V in new coordinates is given by $\varphi(z_1, \dots, z_q) = (z_1, \dots, z_q, 0, \dots)$.*

Proof. Take some coordinate chart (V_1, h_1) in a neighborhood of b in \mathcal{X} . Choose a frame $\{v_i\}_{i=1}^\infty$ in l^2 in such a way that:

- (a) $d_a\varphi(e_i) = v_i, i = 1, \dots, q$, for the standard basis e_1, \dots, e_q of \mathbb{C}^q ;
- (b) $\text{span}\{v_1, \dots, v_q\}$ is orthogonal to $\text{span}\{v_{q+1}, \dots\}$.

Let $z' = (z'_1, \dots, z'_q), w' = (w'_{q+1}, \dots)$ be linear coordinates in l^2 which correspond to the frame $\{v_i\}$ and such that $h_1(b) = (a, 0)$ in these coordinates. Represent $\varphi = (\varphi_1, \varphi_2)$ in coordinates (z', w') . One has $\frac{\partial(z'_1, \dots, z'_q)}{\partial(z_1, \dots, z_q)}(0) = 1_q$ due to (a). By implicit function theorem there exist neighborhoods $U \ni a$ in \mathbb{C}^q and $V_2 \ni a$ in $L = \text{span}\{v_1, \dots, v_q\}$ such that $\varphi_1 : U \rightarrow V_2$ is a biholomorphism. U can be taken of the form $\{\|z - a\| < \delta\}$ for an appropriate $\delta > 0$.

Therefore $\varphi(U)$ is a graph $w' = \psi(z')$ over V_2 . Make a coordinate change $h_2 : (z', w') \rightarrow (z'' = z', w'' = w' - \psi(z'))$ to get that $(h_2 \circ h_1) \circ \varphi$ has the form $(z \rightarrow (\varphi_1(z), 0))$. Finally make one more coordinate change $(z'', w'') \rightarrow (\tilde{z} = \varphi_1^{-1}(z''), \tilde{w} = w'')$ to get the final chart (V, h) with $(h \circ \varphi)(z) = (z, 0)$. V' can be taken to have the form $U \times V_3$ where $V_3 = \{\tilde{w}, \|\tilde{w}\| < \delta\}$ for an appropriate $\delta > 0$ (for that one might need to shrink V_2 and therefore U). Lemma 2.1 and therefore the Step 1 are proved. □

Remark 2.2. U was choosed to be a polydisk. Note that it can be chosen to be a cube as well.

Step 2. Trivializaion of the normal bundle. Cover $\varphi(\overline{D^r})$ with a finite collection of coordinated neighborhoods $\{(V_\alpha, h_\alpha)\}_{\alpha=1}^N$ with centers at a_α as in Step 1. Denote by z^α, w^α the corresponding coordinates in $V'_\alpha \subset l^2$. Note that on this stage z^α glues to a global coordinate z on D . Denote by $J_{\alpha,\beta}$ the Jacobian matrix of the coordinate change $(z^\alpha, w^\alpha) = (h_\alpha \circ h_\beta)^{-1}(z^\beta, w^\beta)$. Since $h_\alpha \circ h_\beta^{-1}(z, 0) = (z, 0)$ we see that

$$J_{\alpha,\beta}(z, 0) = \begin{pmatrix} I_q & A_{\alpha,\beta}(z) \\ 0 & B_{\alpha,\beta}(z) \end{pmatrix}. \quad (2.5)$$

By construction the operator valued functions $B_{\alpha,\beta}$ form a multiplicative cocycle, i.e., they are the transition functions of an appropriated vector l^2 -bundle over $\overline{D^r}$. This bundle is trivial and therefore one can find holomorphic operator valued functions $B_\alpha : U_\alpha \rightarrow \text{End}(l^2)$ such that $B_{\alpha,\beta} = B_\alpha \circ B_\beta^{-1}$ on $U_\alpha \cap U_\beta$.

Make a coordinate change in V_α as follows: $\tilde{z}^\alpha = z^\alpha$ (i.e., still $\tilde{z}^\alpha = z$ for all α) and $\tilde{w}^\alpha = B_\alpha(z^\alpha)^{-1}w^\alpha$. Then in new coordinates the Jacobian matrix of the coordinate change (when restricted to $w^\beta = 0$) will have the form

$$J_{\alpha,\beta}(z; 0) = \begin{pmatrix} I_q & A_{\alpha,\beta}(z) \\ 0 & I_\infty \end{pmatrix}. \quad (2.6)$$

Operator valued functions $A_{\alpha,\beta} \in \text{Hol}(U_\alpha \cap U_\beta, \text{Hom}(l^2, \mathbb{C}^q))$ will satisfy an additive cocycle condition. We can resolve this cocycle, i.e., find $A_\alpha \in \text{Hol}(U_\alpha, \text{Hom}(l^2, \mathbb{C}^q))$ such that $A_{\alpha,\beta} = A_\alpha - A_\beta$ on $U_\alpha \cap U_\beta$. Note that in all these cases we do not need to refine the covering $\{U_\alpha\}$ because it is already acyclic. Performing the coordinate change: $\hat{z}^\alpha = \tilde{z}^\alpha - A_\alpha(\tilde{z}^\alpha)\tilde{w}^\alpha$ and $\hat{w}^\alpha = \tilde{w}^\alpha$ we obtain new coordinates (denote them back as (z^α, w^α)) on V_α such that the coordinate changes satisfy:

- i) $z^\alpha = z^\beta$ when $w^\beta = 0$;
- ii) and

$$J_{\alpha,\beta}(z, 0) = \begin{pmatrix} I_q & 0 \\ 0 & I_\infty \end{pmatrix}. \quad (2.7)$$

Denote by $V = \bigcup_{\alpha=1}^N V_\alpha$ the neighborhood of $\Phi^r = \varphi(\overline{D^r})$ thus obtained. We still denote by $h_\alpha : V_\alpha \rightarrow V'_\alpha \subset l^2$ the corresponding coordinate charts. The coordinate changes have the form

$$\begin{cases} z^\alpha = z^\beta + \sum_{n=2}^{\infty} A_{\alpha,\beta}^n(z^\beta)(w^\beta), \\ w^\alpha = w^\beta + \sum_{n=2}^{\infty} B_{\alpha,\beta}^n(z^\beta)(w^\beta), \end{cases} \quad (2.8)$$

where $A_{\alpha,\beta}^n$ (resp. $B_{\alpha,\beta}^n$) are \mathbb{C}^q -valued (resp. l^2 -valued) holomorphic in z on $U_\alpha \cap U_\beta$ functions that are homogeneous polynomials of degree n in w -s. They satisfy the additive cocycle condition (for every fixed n). Moreover, convergence of these series means that coefficients satisfy the estimates

$$\limsup_{n \rightarrow \infty} \|A_{\alpha,\beta}^n\|_{L^\infty(U_\alpha \cap U_\beta)} \leq \frac{1}{\varepsilon} \quad (2.9)$$

with some $\varepsilon > 0$ independent of α and β . Here $\|A_{\alpha,\beta}^n\|_{L^\infty(U_\alpha \cap U_\beta)} = \sup\{\|A_{\alpha,\beta}^n(z^\beta)\| : z^\beta \in U_\alpha \cap U_\beta\}$. The same for $B_{\alpha,\beta}^n$. Here we use the fact that our coordinate changes are bounded holomorphic mappings.

Step 3. *Solution of an additive Cousin problem with estimates.* For every fixed n we solve $A_{\alpha,\beta}^n = A_\alpha^n - A_\beta^n$ and $B_{\alpha,\beta}^n = B_\alpha^n - B_\beta^n$ with estimates:

$$\begin{cases} \|A_\alpha^n\|_{L^\infty(\overline{D^{r/2}})} \leq C \cdot \max_{\alpha,\beta} \{\|A_{\alpha,\beta}^n\|_{L^\infty(D^r)}\}, \\ \|B_\alpha^n\|_{L^\infty(\overline{D^{r/2}})} \leq C \cdot \max_{\alpha,\beta} \{\|B_{\alpha,\beta}^n\|_{L^\infty(D^r)}\}, \end{cases} \quad (2.10)$$

where C is some constant. To prove (2.10) take some partition of 1 subordinate to the covering $\{U_\alpha\}$ of $\overline{D^r}$ and solve the additive problem in smooth functions, i.e., set

$$\tilde{A}_\alpha^n := \sum_{\beta} \rho_\beta A_{\alpha,\beta}^n.$$

And then put $\hat{A}_\alpha^n = \bar{\partial} \tilde{A}_\alpha^n$. This gives us a globally defined $\bar{\partial}$ -closed $(0,1)$ -form, call it \hat{A}^n (all are homogeneous polynomials). Estimate for the sup norm of \hat{A}^n is immediate. Using Theorem 2.1 we can solve $\bar{\partial} A^n = \hat{A}^n$ on D^r with uniform estimates on \bar{D} . All what is left is to set $A_\alpha^n = \tilde{A}_\alpha^n - A^n$. The rest is obvious.

Step 4. In every chart V'_α make the coordinate change:

$$\begin{cases} \tilde{z}^\alpha = z^\alpha - \sum_{n=2}^{\infty} A_\alpha^n(z^\alpha)(w^\alpha) \\ \tilde{w}^\alpha = w^\alpha - \sum_{n=2}^{\infty} B_\alpha^n(z^\alpha)(w^\alpha). \end{cases} \quad (2.11)$$

Remark that since we have that

$$\limsup_{n \rightarrow \infty} (\|A_\alpha^n\|_{L^\infty(\overline{D^{r/2}})})^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} (C \max_{\alpha,\beta} \{\|A_{\alpha,\beta}^n\|_{L^\infty(D^r)}\})^{\frac{1}{n}} \leq \frac{1}{\varepsilon},$$

we see that power series in (2.11) converge for $\|w^\alpha\| < \varepsilon$. Now we get that the coordinate changes in new coordinates are identity mappings, i.e., V_α glue to a global neighborhood of Φ in \mathcal{X} biholomorphic to a neighborhood of $\bar{D} \subset l_q^2$ in l^2 .

□

2.3. Hartogs manifolds. Recall that a Hartogs figure of bidimension $(q,1)$ is the following domain in \mathbb{C}^{q+1}

$$\begin{aligned} H_q^1(r) &:= \{(z', z_{q+1}) \in \mathbb{C}^{q+1} : \|z'\| < 1, |z_{q+1}| < r \text{ or } 1-r < \|z'\| < 1, |z_{q+1}| < 1\} = \\ &= (B^q \times \Delta(r)) \cup (A_{1-r,1}^q \times \Delta). \end{aligned} \quad (2.12)$$

Here $z' = (z_1, \dots, z_q)$ are coordinates in \mathbb{C}^q .

Let \mathcal{X} be a complex Hilbert manifold.

Definition 2.1. We say that \mathcal{X} is q -Hartogs if for every $r > 0$ every holomorphic mapping $f : H_q^1(r) \rightarrow \mathcal{X}$ extends to a holomorphic map $\tilde{f} : B^q \times \Delta \rightarrow \mathcal{X}$.

1-Hartogs manifolds will be called simply Hartogs. A closed submanifold of a q -Hartogs manifold is q -Hartogs. If, for example, $P(z) = \sum_{k=1}^{\infty} z_k^2$ then the manifold $\mathcal{X} = \{z \in l^2 : P(z) = 1\}$ is Hartogs and analogously any \mathcal{X} , which is defined as a non singular zero set of finitely many continuous polynomials in l^2 is Hartogs.

The following statement can be proved along the same lines as Proposition 4 in [Iv1].

Proposition 2.1. (a) *If \mathcal{X} is a Hilbert manifold and \mathcal{Y} is some unramified cover of \mathcal{X} then \mathcal{X} and \mathcal{Y} are q -Hartogs or not simultaneously.*

(b) *If the fiber \mathcal{F} and the base \mathcal{B} of a complex Hilbert fiber bundle $(\mathcal{E}, \mathcal{F}, \pi, \mathcal{B})$ are q -Hartogs then the total space \mathcal{E} is also q -Hartogs.*

If, for example, $\Lambda = \text{span}_{\mathbb{Z}}\{e_1, ie_1, e_2, ie_2, \dots\}$ is the integer lattice in l^2 , then $\mathbb{T}^{\infty} := l^2/\Lambda$ is Hartogs.

The following statement is immediate via the Docquier-Grauert characterization of Stein domains over Stein manifolds, see [DG].

Proposition 2.2. *If \mathcal{X} is a Hartogs manifold then for every domain D over a Stein manifold every holomorphic mapping $f : D \rightarrow \mathcal{X}$ extends to a holomorphic mapping $\hat{f} : \hat{D} \rightarrow \mathcal{X}$ of the envelope of holomorphy \hat{D} of D to \mathcal{X} .*

2.4. Infinite dimensional Hartogs figures. As we told in the Introduction Hartogs manifolds have better extension properties than it is postulated in their definition. Let us make a first step in proving this. For positive integers q, n and real $0 < r < 1$ we call a Hartogs figure of bidimension (q, n) (or q -concave Hartogs figure) the following set in \mathbb{C}^{q+n}

$$H_q^n(r) := B^q \times B^n(r) \cup A_{1-r,1}^q \times B^n. \quad (2.13)$$

The envelope of holomorphy of $H_q^n(r)$ is $B^q \times B^n$. If holomorphic/meromorphic maps with values in Hilbert manifold \mathcal{X} holomorphically/meromorphically extend from $H_q^n(r)$ to $B^q \times B^n$ we shall say that \mathcal{X} possesses a holomorphic/meromorphic extension property in bidimension (q, n) . Let us prove that holomorphic extendability in bidimension $(q, 1)$ implies the holomorphic extendability in all bidimensions (p, m) with $p \geq q$ and $m \geq n$. The proof follows the lines of Lemmas 2.2.1 and 2.2.2 in [Iv3] (i.e., the finite dimensional case) and therefore we shall give only the detailed proof of the first step, pointing out at what place one needs the Royden's Lemma for Hilbert manifolds. All other steps are just repetitions of the first one and will be only sketched.

Remark 2.3. When working with Hartogs figures one can replace B^q (resp. B^n) in their definitions by the polydisks of the corresponding dimension. This will be not the case when we shall switch to B^{∞} - the unit ball in l^2 .

Proposition 2.3. *If a complex Hilbert manifold \mathcal{X} possesses a holomorphic extension property in bidimension (q, n) for some $q, n \geq 1$ then \mathcal{X} possesses this property in every bidimension (p, m) with $p \geq q, m \geq n$.*

Proof. We shall prove this by induction working with the polydisk model of the Hartogs figure, somehow ambiguously using for it the same notation:

$$H_q^n(r) := (\Delta^q \times \Delta^n(r)) \cup (A_{1-r,1}^q \times \Delta^n),$$

where, this time, $A_{1-r,1}^q := \Delta^q \setminus \bar{\Delta}^q(1-r)$. Let us increase q first. Remark that

$$H_{q+1}^n(r) \supset \bigcup_{z_{q+1} \in \Delta(r)} H_q^n(r) \times \{z_{q+1}\},$$

and therefore a holomorphic mapping $f : H_{q+1}^n(r) \rightarrow \mathcal{X}$ extend along these slices to a map $\tilde{f} : \Delta^q \times \Delta(r) \times \Delta^n \rightarrow \mathcal{X}$ - we respect here the order of variables: $z \in \mathbb{C}^q$, $z_{q+1} \in \mathbb{C}$, $w \in \mathbb{C}^n$ and use the notation $Z = (z, z_{q+1}, w)$ for these coordinates. We need to prove is that this extension \tilde{f} is continuous. Let a sequence $Z_k = (z_1^k, \dots, z_q^k, z_{q+1}^k, w_1^k, \dots, w_n^k)$ converge to $Z_0 = (z_1^0, \dots, z_q^0, z_{q+1}^0, w_1^0, \dots, w_n^0) \in \Delta^{q+1+n}$. Take $0 < R < 1$ such that $\|z^0\|, |z_{q+1}^0|, \|w^0\| < R$ and the same is true for $\|z^k\|, |z_{q+1}^k|, \|w^k\|$ with k big enough. Consider the following imbedded $(q+n)$ -disk Φ^0 in the Hilbert manifold $\mathcal{Y} := \Delta^{q+1+n} \times \mathcal{X}$:

$$\varphi_0(z, w) = \left\{ (z, z_{q+1}^0, f(z, z_{q+1}^0, w)) : z \in \Delta^q(R), w \in \Delta^n(R) \right\}$$

i.e., Φ^0 is the graph of the restriction of f to the $(q+n)$ -disk $\Delta^q(R) \times \{z_{q+1}^0\} \times \Delta^n(R) \subset \Delta^{q+1+n}$.

By Royden's Lemma there exists a neighborhood V of $\Phi^0 = \overline{\varphi(\Delta^q(R) \times \{z_{q+1}^0\} \times \Delta^n(R))}$ in \mathcal{Y} biholomorphic to $V' = \Delta^{n+1} \times B^\infty \subset l^2$. Remark that obviously for z_{q+1} close to z_{q+1}^0 images of $\Delta^q(R) \times \{z_{q+1}\} \times \Delta^n(R)$ under f belong to V by maximum principle. Therefore the question of continuity is reduced to the case of f taking its values in $V' \subset l^2$, and there it is obvious. Therefore f is holomorphically extended to $\Delta^q \times \Delta(r) \times \Delta^n$ and therefore is actually holomorphic on

$$\Delta^q \times \Delta(r) \times \Delta^n \cup H_{q+1}^n(r) = H_q^1(r) \times \Delta^n.$$

Repeating the same argument as above we extend f to Δ^{q+1+n} .

Increase of n follows the same lines and can be fulfilled also in two steps. Set

$$E_q^{n+1}(r) = H_q^n(r) \times \Delta,$$

and remark that $H_q^{n+1}(r) \subset E_q^{n+1}(r)$. Extend f from $H_q^{n+1}(r)$ to $E_q^{n+1}(r)$ exactly as above. Then extend f from $E_q^{n+1}(r)$ to Δ^{q+n+1} again in the same way. □

Remark 2.4. In [Iv3] it was shown with an example that meromorphic extendability in bidimension $(1,1)$ doesn't imply that in bidimensions neither $(1,2)$ (which is not surprising) but also not in bidimension $(2,1)$, even if the image manifold is finite dimensional.

Now we shall consider the following infinite dimensional analog of the Hartogs figure.

Definition 2.2. A q -concave Hartogs figure in l^2 is the following open set

$$\begin{aligned} H_q^\infty(r) &:= \{(z', z'') \in l^2 : \|z'\| < 1, \|z''\| < r \text{ or } 1-r < \|z'\| < 1, \|z''\| < 1\} = \\ &= B^q \times B^\infty(r) \cup A_{1-r,1}^q \times B^\infty, \end{aligned} \quad (2.14)$$

where $0 < r < 1$.

Proposition 2.4. Let \mathcal{X} be a q -Hartogs Hilbert manifold. Then for every $r > 0$ every holomorphic mapping $f : H_q^\infty(r) \rightarrow \mathcal{X}$ extends to a holomorphic mapping $f : B^q \times B^\infty \rightarrow \mathcal{X}$.

Proof. Recall that we identify \mathbb{C}^q with $l_q^2 := \text{span}\{e_1, \dots, e_q\} \subset l^2$. For a unit vector $v \in l^2$ orthogonal to \mathbb{C}^q set $L_v := \text{span}\{e_1, \dots, e_q, v\}$. Remark that $L_v \cap H_q^\infty(r) = H_q^1(r)$ and therefore given a holomorphic mapping $f : H_q^\infty(r) \rightarrow \mathcal{X}$ its restriction to $L_v \cap H_q^\infty(r)$ holomorphically extends onto $L_v \cap (B^q \times B^\infty)$. We conclude that for every line $\langle v \rangle \perp \mathbb{C}^q$ the restriction $f|_{L_v}$ holomorphically extends onto $L_v \cap (B^q \times B^\infty)$, giving us an extension \tilde{f} of f onto B^∞ . This extension is correctly defined because for unit vectors $v \neq w$ orthogonal to \mathbb{C}^q the spaces L_v, L_w intersect only by \mathbb{C}^q . The proof of the continuity of \tilde{f} is literally the same as in Proposition 2.3. What is left to prove is that this extension is Gâteaux differentiable.

Take some $z^0 \in B^q \times B^\infty$ and fix some direction v at z^0 . Let $l := \{z^0 + tv : t \in \mathbb{C}\}$ be the line through z^0 in direction v . Find (at most) two vectors v_1, v_2 such that $e_1, \dots, e_q, v_1, v_2$ is the orthonormal basis of the subspace L containing \mathbb{C}^q, z^0 and l . Our extended map \tilde{f} is holomorphic on $L \cap (B^q \times B^\infty) = B^q \times B^2$ by Proposition 2.3 and therefore is differentiable in the direction of v at z^0 , i.e., is Gâteaux differentiable. \square

Remark 2.5. One can also consider the following version of an infinite dimensional Hartogs figure:

$$H_\infty^\infty(r) = B^\infty \times B^\infty(r) \cup (B^\infty \setminus \bar{B}^\infty(1-r)) \times B^\infty. \quad (2.15)$$

Analogously to the proof of Proposition above one can prove that holomorphic maps from $H_\infty^\infty(r)$ to q -Hartogs Hilbert manifolds extend onto $B^\infty \times B^\infty$ (whatever q is).

2.5. Continuity Principle for Hilbert manifolds. Our goal now is to prove that holomorphic mappings with values in Hilbert-Hartogs manifolds possess very strong extension properties. Namely for them a certain "infinite dimensional" version of the classical Continuity Principle is valid. In what follows convergence of analytic q -disks $\Phi_k = \varphi_k(\bar{D}_k)$ to an analytic q -disk $\Phi = \varphi(\bar{D})$ will be understood as convergence of pseudoconvex domains \bar{D}_k to \bar{D} in \mathcal{C}^2 -sense, and uniform convergence of φ_k to φ on some neighborhood of \bar{D} .

Likewise with the families of analytic q -disks. By a continuous family of strongly pseudoconvex domains in \mathbb{C}^q we understand a domain $D \subset I \times \mathbb{C}^q$ such that for the canonical projection $\pi : I \times \mathbb{C}^q \rightarrow I$ the preimages $D_t := \pi^{-1}(t) \cap D$ are strongly pseudoconvex for all $t \in I$ and continuously depend on t in \mathcal{C}^2 -topology. We shall always suppose that this deformation can be extended to a neighborhood \tilde{D} of the closure of D in $I \times \mathbb{C}^q$ and we set $\tilde{D}_t := \pi^{-1}(t) \cap \tilde{D}$ with the same assumptions.

Definition 2.3. By a continuous family of analytic q -disks in a Hilbert manifold \mathcal{X} we understand a continuous mapping $\varphi : \tilde{D} \rightarrow \mathcal{X}$ which is holomorphic on \tilde{D}_t for every $t \in [0, 1]$.

We shall denote $\varphi(t, \cdot)$ as φ_t and the image $\varphi_t(\bar{D}_t)$ as Φ_t . In practice, most commonly one considers the family $D = I \times B^q$ or $I \times \Delta^q$.

Remark 2.6. Δ^q is not strictly pseudoconvex. But, since we suppose that all our analytic disks are actually defined in a neighborhoods of corresponding closures, we can replace $D = I \times \Delta^q$ by some \tilde{D} close to D such that all \tilde{D}_t are strictly pseudoconvex and work with it.

Let \mathcal{Y} be an another complex Hilbert manifold. The following definition gives us nothing but the usual notion of the analytic continuation along a continuous family of analytic disks.

Definition 2.4. *Let for every $t \in [0, 1]$ be given a connected neighborhood $V_t \supset \Phi_t$ in \mathcal{X} and a holomorphic mapping $f_t : V_t \rightarrow \mathcal{Y}$ such that for every t_1, t_2 close enough $V_{t_1, t_2} := V_{t_1} \cap V_{t_2}$ is connected and $f_{t_1}|_{V_{t_1, t_2}} = f_{t_2}|_{V_{t_1, t_2}}$. In this case shall we say that $\{f_t\}_{t \in [0, 1]}$ is an analytic continuation along $\{\Phi_t\}_{[0, 1]}$ of an analytic mapping $f := f_0$.*

We are ready to state the main result of this paper.

Theorem 2.4. (Continuity Principle) *Let \mathcal{U} be an open subset in a Hilbert manifold \mathcal{X} and let \mathcal{Y} be a q -Hartogs Hilbert manifold. Suppose we are given a continuous family of analytic q -disks $\Phi_t = \varphi_t(\bar{D}_t)$ in \mathcal{X} such that*

- i) $\varphi_0(\bar{D}_0) \subset \mathcal{U}$;
- ii) $\varphi_t(\partial D_t) \subset \mathcal{U}$ for all $t \in [0, 1]$.

Then every holomorphic mapping $f : \mathcal{U} \rightarrow \mathcal{Y}$ analytically extends along $\{\Phi_t\}_{[0, 1]}$.

Proof. The set T of those $t_0 \in [0, 1]$ up to which f is analytically extendable along $\{\Phi_t\}_{[0, t_0]}$ is clearly open. Let us prove that T is also closed. Take $t_1 = \sup \{t_0 : t_0 \in T\}$. Due to our assumptions we can suppose that the domain of definition of all φ_t with t close to t_1 is the same, i.e., is D_{t_1} . In the Hilbert manifold $\mathbb{C}^q \times \mathcal{X}$ consider an imbedded analytic disk $\bar{B}_{t_1} = \{(\lambda, \varphi_{t_1}(\lambda)) : \lambda \in \bar{D}_{t_1}\}$ as well as imbedded analytic disks $\bar{B}_t = \{(\lambda, \varphi_t(\lambda)) : \lambda \in \bar{D}_t\}$ approaching \bar{B}_{t_1} .

Extend each mapping f_t from $V_t \supset \Phi_t$ to $\mathbb{C}^q \times V_t$ not depending on the first coordinate as well as extend f to $\mathbb{C}^q \times \mathcal{U}$. Due to Royden's Lemma 2.3 there exists a biholomorphic mapping h between a neighborhood V of \bar{B}_{t_1} in $\mathbb{C}^q \times \mathcal{X}$ and $\bar{D}_{t_1} \times B^\infty$ sending \bar{B}_{t_1} to $\bar{D}_{t_1} \times \{0\}$. For t close to t_1 we have that $\bar{B}_t \subset V$ and therefore $h(\bar{B}_t)$ is a graph of some holomorphic $\psi_t : \bar{B}_{t_1} \rightarrow B^\infty$ with ψ_t converging uniformly to zero as $t \nearrow t_1$.

Take t_0 sufficiently close to t_1 and make a coordinate change in (a neighborhood of) $\bar{D}_{t_1} \times B^\infty$ as follows: $h_0 : (z, w) \rightarrow (z, \psi_{t_0}(w))$. Mapping $f \circ h^{-1} \circ h_0^{-1}$ is defined and holomorphic on the Hartogs figure $H_q^\infty(r)$ for an appropriate $r > 0$. Theorem follows now from Proposition 2.4.

□

3. LOOP SPACES OF HARTOGS MANIFOLDS ARE HILBERT-HARTOGS

3.1. Loop spaces of complex manifolds. Fix a compact, connected, n -dimensional real manifold (with boundary or not) S and let us following [L1] describe the natural complex Hilbert structure on the Sobolev manifold $W^{k, 2}(S, X)$ of $W^{k, 2}$ -mappings of S into a complex manifold X . To speak about Sobolev spaces it is convenient to suppose that X is imbedded into some \mathbb{R}^N . If X is not compact, we suppose that this embedding is proper. For the following basic facts about Sobolev spaces we refer to [T].

- i) $f \in W^{k, 2}(\mathbb{R}^n) \iff (1 + \|\xi\|)^k \hat{f} \in L^2(\mathbb{R}^n)$, where \hat{f} is the Fourier transform of f .

Moreover this correspondence is an isometry by the Plancherel identity. One defines then for any positive s the space $W^s(\mathbb{R}^n) = \{f : (1 + \|\xi\|)^s \hat{f} \in L^2(\mathbb{R}^n)\}$.

- ii) If $s \geq \frac{n}{2} + \alpha$ with $0 < \alpha < 1$ then $W^s(\mathbb{R}^n) \subset C^\alpha(\mathbb{R}^n)$ and this inclusion is a compact operator. In particular $W^{n, 2}(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$.

- iii) If $s > \frac{n}{2} + k$ for a positive integer k , then $W^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$.

iv) If $0 \leq s < \frac{n}{2}$ then $W^s(\mathbb{R}^n) \subset L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$.

From (ii) one easily derives that if $f, g \in W^{n,2}(\mathbb{R}^n)$ then $fg \in W^{n,2}(\mathbb{R}^n)$. This enables to define correctly a $W^{k,2}$ - vector bundle over an n -dimensional real manifold provided $k \geq n$, and the latter will be always assumed from now on. By that we mean that the transition functions of the bundle are in $W^{k,2}$. Take now $f \in W^{k,2}(S, X)$. Note that by (ii) such f is Hölder continuous. Consider the pullback f^*TX as a complex Sobolev bundle over S . A neighborhood V_f of the zero section of $f^*TX \rightarrow S$ is an open set of the complex Hilbert space $W^{k,2}(S, f^*TX)$ of Sobolev sections of the pullback bundle. This V_f can be naturally identified with a neighborhood of f in $W^{k,2}(S, X)$ thus providing a structure of complex Hilbert manifold on $W^{k,2}(S, X)$. We denote this manifold as $\mathcal{W}_{S,X}^k$.

Another way to understand this complex structure on $W^{k,2}(S, X)$ is to describe what are analytic disks in $W^{k,2}(S, X)$.

Lemma 3.1. *Let D and X be finite dimensional complex manifolds and let S be an n -dimensional compact real manifold. A mapping $F : D \times S \rightarrow X$ represents a holomorphic map from D to $W^{k,2}(S, X)$ (denoted by the same letter F) if and only if the following holds:*

- i) *for every $s \in S$ the map $F(\cdot, s) : D \rightarrow X$ is holomorphic;*
- ii) *for every $z \in D$ one has $F(z, \cdot) \in W^{k,2}(S, X)$ and this correspondence $D \ni z \rightarrow F(z, \cdot) \in W^{k,2}(S, X)$ is continuous with respect to the Sobolev topology on $W^{k,2}(S, X)$ (and standard topology on D).*

For the proof we refer to [L1]. Let us prove the main result of this section:

Theorem 3.1. *Let X be a q -Hartogs manifold and S be an n -dimensional real compact manifold. Then the Hilbert manifold $\mathcal{W}_{S,X}^k = W^{k,2}(S, X)$ is also q -Hartogs.*

Proof. Let $F : H_q^1(r) \rightarrow W^{k,2}(S, X)$ be a holomorphic map. We represent this map as a mapping $F : H_q^1(r) \times S \rightarrow X$ possessing properties i) , ii) of the lemma above. From the fact that X is q -Hartogs we get immediately that for every fixed $s \in S$ the mapping $F(\cdot, s) : H_q^1(r) \rightarrow X$ extends holomorphically to $\tilde{F}(\cdot, s) : B^q \times \Delta \rightarrow X$ and we get a mapping $\tilde{F} : (B^q \times \Delta) \times S \rightarrow X$. All what left is to prove that for every fixed $z \in B^q \times \Delta$ one has $\tilde{F}(z, \cdot) \in W^{k,2}(S, X)$ and that this correspondence is continuous. The question is local, so using a partition of unity on S we can reduce it to the following lemma. Denote by $W_0^{k,2}(\mathbb{R}^n, X)$ the space of Sobolev maps from \mathbb{R}^n to X with support contained in the closed unit ball \bar{B}^n of \mathbb{R}^n .

Lemma 3.2. *Consider a mapping $F : (B^q \times \Delta) \times \bar{B}^n \rightarrow X$ to a q -Hartogs manifold X such that:*

- i) *for every $z \in H_q^1(r)$ the map $F(z, \cdot)$ is in $W_0^{k,2}(\mathbb{R}^n, X)$;*
- ii) *$z \rightarrow F(z, \cdot)$ is a continuous map from $H_q^1(r)$ to $W_0^{k,2}(\mathbb{R}^n, X)$.*
- iii) *for every $s \in \bar{B}^n$ $F(\cdot, s)$ is holomorphic on $B^q \times \Delta$.*

Then for every $z \in B^q \times \Delta$ the map $F(z, \cdot)$ is in $W_0^{k,2}(\mathbb{R}^n, X)$ and correspondence $z \rightarrow F(z, \cdot)$ is a continuous map from $B^q \times \Delta$ to $W_0^{k,2}(\mathbb{R}^n, X)$.

Proof. From the uniqueness property of holomorphic functions we get immediately that for every $z \in \Delta \times B^q$ one has $\text{supp} F(z, \cdot) \subset \bar{B}^n$. Shrinking $H_q^1(r)$ if necessary and refining our partition of unity we can suppose that $F(H_q^1(r) \times \bar{B}^n) \subset \mathbb{C}^N$ where $N = \dim X$. The

mapping $H^q(r) \ni z \rightarrow F(z, \cdot) \in W_0^{k,2}(\mathbb{R}^n, \mathbb{C}^N)$ is holomorphic and therefore holomorphic is the map $H^q(r) \ni z \rightarrow (1 + \|\xi\|)^k \hat{F}(z, \cdot) \in L^2(\mathbb{R}^n, \mathbb{C}^N)$. The latter holomorphically extends onto $B^q \times \Delta$ and Hilbert norm of a holomorphic Hilbert-valued mapping satisfies the maximum modulus principle, i.e., in particular it will continuously depend on z .

Therefore the lemma and the theorem are proved. \square

The following statement gives us one more example of open sets $\mathcal{U} \subsetneq \hat{\mathcal{U}}$ in Hilbert manifold such that holomorphic mappings extend from \mathcal{U} to $\hat{\mathcal{U}}$ (the previous one was $H_q^\infty(r) \subsetneq B^q \times B^\infty$). It shows that $W^{k,2}(S, B^q \times B^n)$ is the “envelope of holomorphy” of $W^{k,2}(S, H_q^n(r))$.

Theorem 3.2. *Let \mathcal{X} be a q -Hartogs Hilbert manifold. Then every holomorphic map $F : W^{k,2}(S, H_q^n(r)) \rightarrow \mathcal{X}$ extends to a holomorphic map $\tilde{F} : W^{k,2}(S, B^q \times B^n) \rightarrow \mathcal{X}$.*

Proof. Let $f = (f^q, f^n) : S \rightarrow B^q \times B^n$ be an element of $W^{k,2}(S, B^q \times B^n)$. Consider the following analytic q -disk in $W^{k,2}(S, B^q \times B^n)$

$$\varphi : (z, s) \in B^q \times S \rightarrow (h_{f^q(s)}(z), f^n(s)), \quad (3.1)$$

where h_a is an automorphism of B^q interchanging a and 0 . $\Phi = \varphi(\bar{B}^q)$ is clearly an analytic disk in $W^{k,2}(S, B^q \times B^n)$ satisfying the following properties:

- $\varphi(0, s)$ is our loop f .
- For $z \in \partial B^q$ one has that $\varphi(z, \cdot) : S \rightarrow A_{1-r, 1+r}^q \times B^n$. Therefore $\partial\Phi \subset W^{k,2}(S, H_q^n(r))$.

Consider the family

$$\varphi_t(z, s) = (h_{f^q(s)}(z), t f^n(s)).$$

Then

- $\varphi_0 \subset B^q \times \{0\}$.
- $\varphi_1 = \varphi$.
- For all $t \in [0, 1]$ one has that $\partial\Phi_t \subset W^{k,2}(S, H_q^n(r))$.

By the Continuity Principle of Theorem 2.4 our mapping F extends along $\{\Phi_t\}$ as well as it extends along any path in $W^{k,2}(S, B^q \times B^n)$ with described properties. Finally, since $H_q^n(r)$ and $B^q \times B^n$ are contractible the manifold $W^{k,2}(S, B^q \times B^n)$ is simply connected for any S . \square

Corollary 3.1. *If X is a q -Hartogs manifold then every holomorphic mapping $F : W^{k,2}(S, H_q^n(r)) \rightarrow W^{k,2}(S, X)$ extends to a holomorphic mapping $\hat{F} : W^{k,2}(S, B^q \times B^n) \rightarrow W^{k,2}(S, X)$.*

Example 3.1. Let G be a complex Lie group. Then G is Hartogs, see [ASY]. Therefore the loop space, which is classically denoted as LG is also Hartogs.

3.2. Loop spaces of compact manifolds are almost Hartogs. The aim of this subsection is to explain that one finds Hartogs manifolds more often than one might think of them.

As in [Iv4] we introduced the class \mathcal{G}_q of q -disk convex complex manifolds possessing a strictly positive dd^c -closed (q, q) -form. Sequence $\{\mathcal{G}_q\}_{q=1}^\infty$ is rather exhaustive: \mathcal{G}_q contains all compact manifolds of dimension $q+1$, see subsection 1.5 in [Iv4].

We think that the following statement should be true: *Let X be a complex manifold from the class \mathcal{G}_q . Then*

- i) either X contains a $(q+1)$ -dimensional spherical shell (remark that $X \in \mathcal{F}_q$ implies that $\dim X \geq q+1$);*
- ii) or, X contains an uniruled compact subvariety of dimension q ;*
- iii) or, X is q -Hartogs.*

Remark 3.1. a) This was proved in [Iv4] for $q = 1$ (in fact this particular statement was proved already in [Iv2]), and in [IS] for $q = 2$. In the latter paper we proved almost the assertion stated above (for all q -s), but for holomorphic mappings with zero-dimensional fibers, see Proposition 12 there.

2. For $q = 1$ the item (ii) means just that X contains a rational curve. For $q = 2$ we need to add few explanations to [IS]. We proved there that a meromorphic map from $H_2^1(r)$ to such X meromorphically extends to $\Delta^3 \setminus S$, where S is a complete pluripolar set of Hausdorff dimension zero. If $S \neq \emptyset$ then X contains a spherical shell of dimension 3. Otherwise S is empty. If our map f was in addition holomorphic on $H_2^1(r)$ then the set $I_{\tilde{f}}$ of points of indeterminacy of the extension \tilde{f} can be only discrete and then it is clear that for every $a \in I_{\tilde{f}}$ its full image $\tilde{f}[a]$ contains an uniruled analytic set of dimension two.

From this we obtain the following

Corollary 3.2. *Let X be a compact complex manifold of dimension 2 (resp. of dimension 3). Then either X is one of (i) or (ii) as above or, every generalized loop space $\mathcal{W}_{S,X}^k$ is Hartogs (resp. 2-Hartogs).*

It might be interesting to think about X from this Corollary as being (an unknown) surface of class VII_0^+ or as \mathbb{S}^6 .

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